数理解析研究所講究録
第 1733 巻 2011 年 31-39

Verified eigenvalue evaluation for Laplace operator on arbitrary polygonal domain

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Abstract

For eigenvalue problem of Laplace operator over polygonal domain $\Omega(\subset R^2)$ of arbitrary shape, we proposed an algorithm based on finite element method to bound the leading eigenvalues with indices guaranteed. The algorithm is developed by well use of max-min and min-max principles and newly constructed a priori error estimation for FEM solution. The efficiency of the algorithm is demonstrated by several computation examples.

1 Introduction

The eigenvalue problem of Laplacian has been well investigated in history from various viewpoints. Here, we pay attention to giving accurate bounds for eigenvalues with indices guaranteed. For such a purpose, Lehmann-Goerisch method is well known as a effective way to give sharp bounds for eigenvalues once a quantity ν satisfying $\lambda_k < \nu \le \lambda_{k+1}$ is available, where λ_k denotes the eigenvalues with increasing order on magnitude. To find such a ν is not an easy work. In [8], M. Plum developed *homotopy method* based on operator comparison theorem to give a computable ν. As *base problem* with explicit eigenvalues is necessary when we apply the homotopy method, domain mapping is used to construct the base problem, which brings difficulties to solving problems over general domain.

In this paper, starting from an early work of Birkhoff, de Boor, Swartz and Wendroff [2], we propose a new method to give guaranteed estimation for leading k-th eigenvalue on arbitrary polygonal domain, where the finite element method(FEM) is used to give approximate eigenvalues with computable error bounds. The estimate of λ_k obtained by using sparse domain triangulation will be relatively rough, but it can work as a good candidate of ν mentioned above. Thus, if needed, the bounds can be sharpened by further applying Lehmann-Goerisch method. The method proposed here can deal with three types homogeneous boundary conditions associated with function space: Dirichlet, Neumann and mixed one. To explain the method in a concise way, we only show details for the Dirichlet case. Also, it is possible to extend the method for general elliptic problem.

At the end of this paper, we show computation results on triangle and L-shaped domain.

2 Preliminaries

Let Ω be polygonal domain with arbitrary shape, convex or non-convex. We introduce function space $V = H_0^1(\Omega) = \{v \in H^1(\Omega) | v = 0 \text{ on } \partial\Omega\}$. The notation $\|v\|_{L_2}$ denotes the L_2 norm of $v \in L_2$ and $|v|_{H^k(\Omega)}$ $(k = 1, 2)$ the semi-norms in $H^k(\Omega)$. Let

$\mathcal{T}^h$ be one triangularization of Ω, which has polygon boundary. The variational form of eigenvalue problem is defined as below:

$$\text{Find } \lambda \in R \text{ and } u \in V \text{ s.t. } (\nabla u, \nabla v) = \lambda(u, v), \quad \forall v \in V. \tag{1}$$

The classical continuous piecewise linear finite element(FE) space $V_h \subset V$ will be used as approximation space. The Ritz method is to solve the variational problem in V_h,

$$\text{Find } \lambda^h \in R \text{ and } u_h \in V_h \text{ s.t. } (\nabla u_h, \nabla v_h) = \lambda^h\,(u_h, v_h), \quad \forall v_h \in V_h\,. \tag{2}$$

Supposing the bases of V_h to be $\{\phi_i\}_{i=1}^n$, the problem of (2) is in fact a generalized matrix eigenvalue problem:

$$A^h x = \lambda^h B^h, \quad \text{where } A^h_{i,j} = (\nabla\phi_i, \nabla\phi_j)_{L_2}, B^h_{i,j} = (\phi_i, \phi_j)_{L_2}\,. \tag{3}$$

The eigenvalues λ^h_k can be evaluated accurately by applying verified computations, c.f., [1, 7, 9]. Denote by $\{\lambda_i, u_i\}$ (resp. $\{\lambda^h_i, u^h_i\}$) the eigenpairs of (1) (resp. (2)) with eigenfunction being orthogonally normalized under L_2-norm. These eigenpairs are just the stationary values and critical points of Rayleigh quotient on space V (resp. V_h):

$$R(u) := (\nabla u, \nabla u)/(u, u)\,. \tag{4}$$

Since an upper bound for λ_i as $\lambda_i \le \lambda^h_i$ is easy to obtain from min-max principle, we will pay attention to find satisfactory lower bounds for eigenvalues. The eigenfunction estimation will not be discussed here.

Let's introduce two constants $C_{i,h}$ $(i = 0, 1)$ to be used later, which are related to function interpolations π_i $(i = 0, 1)$ over triangle element K. For $u \in L_2(K)$, $\pi_0 u$ is constant function s.t.

$$\pi_0 u \equiv \int_K u(x)\,dx \Big/ \int_K 1\,dx\,, \tag{5}$$

and for $u \in H^2(K)$, $\pi_1 u$ is linear function s.t.

$$(\pi_1 u)\,(x) = u(x) \text{ on each vertex of } K\,. \tag{6}$$

Global interpolations $\pi_{0,h}$ and $\pi_{1,h}$ are just the extension of π_0 and π_1. Define h by the mesh size and $C_{0,h}$ and $C_{1,h}$ the constants over triangulation $\mathcal{T}_h$,

$$C_{i,h} := \max_{K \in \mathcal{T}^h} C_i(K)/h \quad (i = 0, 1)\,, \tag{7}$$

where

$$C_0(K) := \sup_{v \in H^1(K)\backslash\{0\}} |\pi_0 u - u|_{L_2}/|u|_{H^1}, \quad C_1(K) := \sup_{v \in H^2(K)\backslash\{0\}} |\pi_1 u - u|_{H^1}/|u|_{H^2}\,.$$

3 Lower bound of eigenvalues by adopting min-max and max-min principle

In this section, we will introduce two methods to give lower bound for eigenvalues, all of them adopting computable a priori estimate of Ritz-Galerkin solution of Poisson's problem. Let $u \in H^1_0(\Omega)$ be the solution of following variation problem,

$$(\nabla u, \nabla v) = (f, v) \quad \forall v \in H^1_0(\Omega)\,. \tag{8}$$

The solution $u \in H_0^1(\Omega)$, in meaning of distribution, satisfies the partial differential equation $-\Delta u = f$. Whether u belongs to $H^2(\Omega)$ or not depends on the domain shape.

Let P_h be the orthogonal projection of $u \in V$ into V_h,

$$(\nabla u - \nabla P_h u, \nabla v_h) = 0 \quad \forall v_h \in V_h . \tag{9}$$

We will deduce a computable a apriori error estimate in the form as below,

$$|u - P_h u|_{H^1} \leq M\|f\|_{L_2}, \ \|u - P_h u\|_{L_2} \leq M|u - P_h u|_{H^1} \leq M^2\|f\|_{L_2} , \tag{10}$$

where M is quantity to be evaluated in Section 4. In the following, we will introduce two methods to bound eigenvalues based on this a priori error estimation.

3.1 Birkhoff's method: application of Min-Max principle

Birkhoff, de Boor, Swartz and Wendroff [2] considered eigenvalue problem in form of Rayleigh quotient $R(u) := N(u)/D(u)$, where $N(u)$ and $D(u)$ are quadratic forms of $u \in V$ and $D(u) > 0$ for $u \neq 0$. Suppose $\{\lambda_k, u_k\}$ (resp. $\{\lambda_h^k, u_k^h\}$) the stationary values and critical points of $R(u)$ on space V (resp. V_h), with increasing order on λ_k (resp. λ_h^k). Birkhoff et al deduced an estimate for λ_k^h by applying Min-Max principle:

Theorem 1. *Given any $v_1^h, v_2^h, \cdots, v_k^h \in V_h(\subset V)$ satisfying $\sum_{i=1}^k D(v_i^h - u_i) < 1$, we have, for $k \geq 1$,*

$$\lambda_k \leq \lambda_h^k \leq \lambda_k + \left(\sum_{i=1}^k N(v_i^h - u_i)\right) \Bigg/ \left(1 - \left(\sum_{i=1}^k D(v_i^h - u_i)\right)^{1/2}\right)^2 \tag{11}$$

For model problem of Laplacian in (1), $N(u) = (\nabla u, \nabla u)$ and $D(u) = (u, u)$. It is natural to select $v_i^h = P_h u_i$ $(i = 1, \cdots, k)$ for each eigenfunction u_i, and apply the error estimate of (10):

$$|u_i - P_h u_i|_{H^1} \leq M\|\Delta u_i\|_{L_2} = M\lambda_i\|u_i\|_{L_2} = M\lambda_i .$$
$$\|u_i - P_h u_i\|_{L_2} \leq M|u_i - P_h u_i|_{H^1} \leq M^2\lambda_i .$$

Thus, we obtain an a priori estimate for λ_k^h.

Theorem 2. *Let λ_k and λ_k^h be the ones defined in Section 2. If $1 - M^2(\sum_{i=1}^k \lambda_i^2)^{1/2} > 0$, we have*

$$\lambda_k^h \leq \lambda_k + M^2 \sum_{i=1}^k \lambda_i^2 \Bigg/ \left(1 - M^2(\sum_{i=1}^k \lambda_i^2)^{1/2}\right)^2 . \tag{12}$$

Define function ϕ_1 on variable t_k with parameters $\{t_1, \cdots, t_{k-1}\}$ as below,

$$\phi_1(t_k; t_1, \cdots, t_{k-1}) := t_k + M^2 \sum_{i=1}^k t_i^2 \Bigg/ \left(1 - M^2(\sum_{i=1}^k t_i^2)^{1/2}\right)^2 \tag{13}$$

Noticing that ϕ_1 is increasing as varible t_k increases, $\phi_1(t_k)$ has increasing inverse function. Therefore, $\phi_1^{-1}(\lambda_k^h; \lambda_1, \cdots, \lambda_{k-1}) \leq \lambda_k$. As $\lambda_i \leq \lambda_i^h$ for $i \geq 1$, we can further see

$$\phi_1^{-1}(\lambda_k^h; \lambda_1^h, \cdots, \lambda_{k-1}^h) \leq \lambda_k .$$

Remark 3.1. *In practical computation, instead of verifying* $1-M^2(\sum_{i=1}^{k}\lambda_i^2)^{1/2}>0$, *we will check a stronger condition* $1-M^2(\sum_{i=1}^{k}\lambda_i^{h^2})^{1/2}>0$ *since* $\lambda_i^h(\geq\lambda_i)$*'s are computable ones.*

Remark 3.2. *Birkhoff, de Boor, Swartz and Wendroff [2] obtained the estimation of form (11) with* V_h *the space constructed by spline functions. By applying the error estimate for spline interpolation, quantitative estimate for eigenvalue problem of 1D Sturm-Liouville system is successfully done. However, it is difficult to apply their method to solve problem on general 2D domain. As a comparison, our estimation in Theorem 2 and 3 can deal with eigenvalue problem with domain of general shape, which inherits advantages from the finite element method.*

3.2 Bounding eigenvalues by adopting Max-Min principle

Theorem 3 (Liu). *Let* $v_1^h,\cdots,v_{k-1}^h$ *be arbitrary functions of* V_h *and* $V_{k-1}:=span\{v_1^h,\cdots,v_{k-1}^h\}$. *Define* $\tilde{\lambda}_k$ *by Rayleigh quotient on* $V_h\cap V_{k-1}^{\perp}$ *(* $V_{k-1}^{\perp}$ *: complement space of* V_{k-1} *in* V*)*

$$\tilde{\lambda}_k=\min_{v_h\in V_h\cap V_{k-1}^{\perp}}\frac{(\nabla v_h,\nabla v_h)}{(v_h,v_h)}.$$

Then, an a posteriori estimate for $\tilde{\lambda}_k$ *is available,*

$$\tilde{\lambda}_k-\lambda_k\leq\left(M\tilde{\lambda}_k\right)^2\Big/\left(1+M^2\tilde{\lambda}_k\right), \tag{14}$$

where M *is the one in (10).*

Proof. From Max-Min principle, we have

$$\lambda_k=\max_{W\subset V,\dim(W)\leq k-1}\ \min_{v\in W^{\perp}}\frac{(\nabla v,\nabla v)}{(v,v)}.$$

Thus, for specified $V_{k-1}:=\text{span}\{v_1^h,\cdots,v_{k-1}^h\}$, a lower bound for λ_k is given as

$$\lambda_k\geq\min_{v\in V_{k-1}^{\perp}}\frac{(\nabla v,\nabla v)}{(v,v)}. \tag{15}$$

For any $v\in V_{k-1}^{\perp}$, $P_hv\in V_h$. Let w_h be arbitrary one in $V_{k-1}(\subset V_h)$. Then $(\nabla v,\nabla w_h)=0$. Thus $(\nabla P_hv,\nabla w_h)=(\nabla v,\nabla w_h)=0$, which implies that $P_hv\in V_h\cap V_{k-1}^{\perp}$. Considering (10) and the definition of $\tilde{\lambda}_k$,

$$\|v\|_{L_2}\leq\|P_hv\|_{L_2}+\|v-P_hv\|_{L_2}\leq\tilde{\lambda}_k^{-1/2}\|\nabla P_hv\|_{L_2}+M\|\nabla(v-P_hv)\|_{L_2}.$$

$$\|v\|_{L_2}^2\leq\left(\tilde{\lambda}_k^{-1}+M^2\right)(\|\nabla P_hv\|_{L_2}^2+\|\nabla(v-P_hv)\|_{L_2}^2)=\left(\tilde{\lambda}_k^{-1}+M^2\right)\|\nabla v\|_{L_2}^2.$$

Hence,

$$\|\nabla v\|_{L_2}^2/\|v\|_{L_2}^2\geq\tilde{\lambda}_k/\left(1+M^2\tilde{\lambda}_k\right)\ \text{ for any } v\in V_{k-1}^{\perp}.$$

The equation (15) tells us,

$$\lambda_k\geq\tilde{\lambda}_k/\left(1+M^2\tilde{\lambda}_k\right). \tag{16}$$

Now, it is trivious to formulate the result in (14). □

Remark 3.3. *The subspace V_{k-1} in Theorem 3 can be taken as the one spanned by first $k-1$ eigenfunction of (2). In this case, $\tilde{\lambda}_k = \lambda_k^h$ and lower bound of λ_k is given as:*

$$\lambda_k^h / \left(1 + M^2 \lambda_k^h\right) \le \lambda_k \le \lambda_k^h \,. \tag{17}$$

It is obvious that the above estimate based on Max-Min principle gives better estimate than the one of (11).

4 A prioiri error estimate for Ritz-Galerkin solution of Poisson's problem

The following section will be devoted to evaluating M appearing in a priori error estimate (10) for projection P_h. The discussion will be divided into two parts, the one with regular solution on convex domain and the one with singular solution on non-convex domain.

4.1 Convex domain

First, we quote a well known result on a priori estimation for Laplacian.

Lemma 4. *[4] Assume Ω is bounded convex polygonal domain in R^2. For $u \in H^2(\Omega) \cap H_0^1(\Omega)$ or $u \in H^2(\Omega)$ and $\partial u / \partial n = 0$ on $\partial\Omega$, let $f := -\Delta u$. Then, we have*

$$|u|_{H^2} \le \|\Delta u\|_{L_2} = \|f\|_{L_2} \,.$$

Theorem 5. *Let Ω be convex polygonal domain and u be the solution of (8). The error estimate for $(u - P_h u)$ is given as*

$$|u - P_h u|_{H^1} \le C_{1,h} h \|f\|_{L_2}, \quad \|u - P_h u\|_{L_2} \le C_{1,h} h |u - P_h u|_{H^1} \le C_{1,h}^2 h^2 \|f\|_{L_2} \,.$$

Thus, we can take $M := C_{1,h} h$ under current assumptions.

Proof. Under the given assumptions, the solution u belongs to $H^2(\Omega)$. By using interpolation error estimate for $\pi_{1,h}$ and the Lemma 4, we have,

$$|u - P_h u|_{H^1} \le |u - \pi_{1,h} u|_{H^1} \le C_{1,h} h |u|_{H^2} \le C_{1,h} h \|f\|_{L_2} \,, \tag{18}$$

where the constant $C_{1,h}$ is the one defined in (7). The L_2-norm error estimation can be easily done by adopting Aubin-Nitsue's technique.

□

4.2 Non-convex domain

To deal with problem on non-convex domain, which has singular solution not belonging to $H^2(\Omega)$, we adopt the hypercircle equation to deduce a computable a priori error estimate. Let W^h be the lowest order Raviart-Thomas FEM space over domain triangulation $\mathcal{T}^h$ and M^h the space of piecewise constant. Also, define subspace of W_h for f_h in M^h, $W_{f_h}^h := \{p_h \in W^h | \operatorname{div} p_h = f_h\}$. Recall the definition of $\pi_{0,h} : L_2(\Omega) \to M^h$ in Section 2,

$$(u - \pi_{0,h} u, v_h) = 0, \quad \forall v_h \in M^h \,.$$

From the definition, we have $\|u\|_{L^2}^2 = \|\pi_{0,h}u\|_{L^2}^2 + \|u - \pi_{0,h}u\|_{L^2}^2$ and

$$\|u - \pi_{0,h}u\|_{L^2} \le C_{0,h}h|u|_{H^1} \quad \text{if } u \in H^1.$$

where $C_{0,h}$ is the constant defined in (7).

Let's introduce a computable quantity κ over finite dimensional spaces:

$$\kappa := \max_{f_h \in M^h \setminus \{0\}} \min_{v_h \in V_h} \min_{p_h \in W^h_{f_h}} \|p_h - \nabla v_h\|_{L_2} / \|f_h\|_{L_2} \tag{19}$$

Lemma 6. *Given $f_h \in M^h$, let $\tilde{u} \in H^1$ and $\tilde{u}_h \in V_h (\subset V)$ be the solutions of variational problems,*

$$(\nabla \tilde{u}, \nabla v) = (f_h, v), \quad \forall v \in V \,. \quad (\nabla \tilde{u}_h, \nabla v_h) = (f_h, v_h), \quad \forall v_h \in V_h \,, \tag{20}$$

respectively. Then we have a computable error estimate as below:

$$|\tilde{u} - \tilde{u}_h|_{H^1} \le \kappa \|f_h\|_{L_2} \,. \tag{21}$$

Proof. From Prager-Synge's theorem, we have, for $\tilde{u}$ in (20) and any $v_h \in V_h$, $p_h \in W^h_{f_h}$, such a *hypercircle equation* holds,

$$\|\nabla \tilde{u} - \nabla v_h\|_{L_2}^2 + \|\nabla \tilde{u} - p_h\|_{L_2}^2 = \|p_h - \nabla v_h\|_{L_2}^2 \,. \tag{22}$$

Thus,

$$\|\nabla \tilde{u} - \nabla v_h\|_{L_2} \le \|\nabla v_h - p_h\|_{L_2}, \quad \forall v_h \in V_h, \forall p_h \in W^h_{f_h} \,. \tag{23}$$

From minimization principle and the definition of κ, we obtain

$$\|\nabla \tilde{u} - \nabla \tilde{u}_h\|_{L_2} \le \min_{v_h \in V_h} \min_{p_h \in W^h_{f_h}} \|p_h - \nabla v_h\|_{L_2} \le \kappa \|f_h\|_{L_2} \,. \tag{24}$$

□

Theorem 7. *For any $f \in L_2(\Omega)$, let $u \in V$ and $u_h \in V_h$ be solution of variational problems*

$$(\nabla u, \nabla v) = (f, v), \quad \forall v \in V, \quad (\nabla u_h, \nabla v_h) = (f, v_h), \quad \forall v_h \in V_h \,. \tag{25}$$

respectively. Introduce quantity $M := \sqrt{C_{0,h}^2 h^2 + \kappa^2}$, where $C_{0,h}$ is the constant defined in (7). Then, we have,

$$|u - u_h|_{H^1} \le M \|f\|_{L_2}, \quad \|u - u_h\|_{L_2} \le M^2 \|f\|_{L_2} \,. \tag{26}$$

Remark 4.1. *The quantity M, independent of f, will decrease when mesh is refined. By theoretical analysis, we can show that M tends to 0 in the same order as the error of linear conforming FEM solution solution.*

Proof. We follow analogous framework with Kikuchi and Saito [6] to finish the proof. Let $\tilde{u}$ and $\tilde{u}_h$ be the ones defined in Theorem 6 with $f_h = \pi_{0,h}f$. The minimization principle leads to $|u - u_h|_{H^1} \le |u - \tilde{u}_h|_{H^1}$. Decomposing $u - \tilde{u}_h$ by $(u - \tilde{u}) + (\tilde{u} - \tilde{u}_h)$, we have

$$|u - u_h|_{H^1} \le |u - \tilde{u}_h|_{H^1} \le |u - \tilde{u}|_{H^1} + |\tilde{u} - \tilde{u}_h|_{H^1} \,. \tag{27}$$

From the definitions of u and $\tilde{u}$, we have, for any $v \in V$

$$(\nabla(u-\tilde{u}), \nabla v) = (f - \pi_{0,h} f, v) = ((I - \pi_{0,h})f, (I - \pi_{0,h})v) \ .$$

Taking v to be $u - \tilde{u}$ and applying the error estimate of interpolation $(I - \pi_{0,h})v$,

$$|u - \tilde{u}|_{H^1} \leq C_{0,h} h \|(I - \pi_{0,h})f\|_{L_2} \ . \tag{28}$$

Substitute (21) and (28) into (27),

$$|u - u_h|_{H^1} \leq C_{0,h} h \|(I - \pi_{0,h})f\|_{L_2} + \kappa \|\pi_{0,h} f\|_{L_2} \leq \sqrt{C_{0,h}^2 h^2 + \kappa^2} \ \|f\|_{L_2} \ .$$

The estimate for $\|u - u_h\|_{L_2}$ can be easily done by applying Aubin-Nitsche's method. □

5 Computation

The computation of quantity κ turns to solving eigenvalue problem of matrix, for which we omit the details but point out that evaluation of κ consumes most of the total computation time. To obtain accurate numerical result, we adopt interval computation arithmetic to do the floating-point computation. The total framework is as below

1) Triangulate the domain Ω and construct finite element space V_h.

2) Solve eigenvalues problem $A^h x = \lambda^h B^h x$ under the bases of V_h.

3) Evaluate quantity M for the mesh and domain.

4) Calculate the lower and upper bounds of λ_k by using (12) or (17) .

In the following we display computation examples on several domains.

5.1 Triangle domain

In case of unit isosoceles right triangle domain, due to the symmetry of specified triangle domain, we can apply reflecting techniques, e.g.,[5], to obtain the explicit eigenpairs as below:

$$\{\lambda = m^2 + n^2, u = \sin m\pi x \sin n\pi y - \sin n\pi x \sin m\pi y\}_{m>n\geq 1} \ .$$

To compare the efficiency of the methods basing on Max-min principle and the Min-max principle, we display the estimates of (12) and (17) in Table 1.

5.2 Computation Results on L-shaped domain

The domain is taken as $\Omega = [0,2] \times [0,2] \setminus [1,2] \times [1,2]$. As a model problem, it has been well explored by many people, e.g., L. Fox, P. Henrici and C. Moler [3]. However, to the author's knowledge, most of the results are given only in the sense of approximate computation. Although our method gives a relatively rough evaluation, it can easily deal with more general domain and the result works as mathematically correct with indices guaranteed.

λ_i	approx.	lower (12)	lower (17)	upper
1	49.348	48.955	48.976	49.553
2	98.696	96.532	97.331	99.633
3	128.305	122.196	125.853	129.729
4	167.783	154.763	163.694	170.312
5	197.392	174.176	192.372	201.577

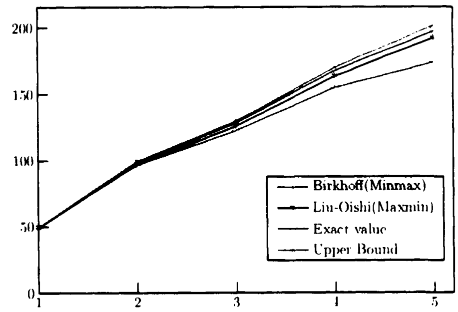


Table 1: Estimates by Min-max(12) and Max-min (17) principle. ($h = 1/32$)

λ_i	exact value	$h = 1/32$		$h = 1/64$	
		lower	upper	lower	upper
1	$5\pi^2 \approx 49.348$	48.976	49.553	49.254	49.400
2	$10\pi^2 \approx 98.696$	97.331	99.633	98.352	98.930
3	$13\pi^2 \approx 128.305$	125.853	129.729	127.687	128.662
4	$17\pi^2 \approx 167.783$	163.694	170.312	166.746	168.413
5	$20\pi^2 \approx 197.392$	192.372	201.577	196.129	198.439

Table 2: Eigenvalue estimates for Laplacian on triangle domain (Dirichlet b.d.c.)

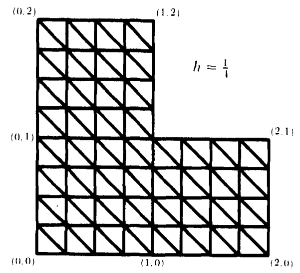
In Table 4, we list the first 5 eigenvalues given by [3] and the verified bounds by our proposed method. The values of κ, C_0^h and M, which are only depending on the mesh, are displayed in Table 3. We can see M tends to zero in order less than 1. Once lower bound for λ_5 is available, we can further apply the Lehmann-Goerisch method to obtain more precise bound for $\lambda_1, \cdots, \lambda_4$. Such a computation, although not verified, has been reported by Yuan and He [10] with very sharp bounds, while the lower bound for λ_5 is obtained in a different way.

6 Conclusion

For the classical eigenvalue problems of Laplace operator over 2-dimensional domain, we have proposed a novel and robust method to give accurate lower and upper bounds for eigenvalues. The method can deal with both convex and non-convex domains with general shape. To apply the Lehmann-Goerisch method for purpose of high precision, we still need pay efforts on constructing base functions over general domain.

Acknowledgement

The authors would like show great appreciation to Prof. M. Plum and Prof. K. Nagatou for fruitful discussion.



h	κ	C_0^h	M	order of M
1/4	0.1466	0.080	0.1668	-
1/8	0.0882	0.040	0.0968	0.786
1/16	0.0538	0.020	0.0574	0.754
1/32	0.0332	0.010	0.0348	0.722

Table 3: Uniform mesh of L-shaped domain and κ values

λ_i	lower bound	approximate	upper bound	relative error
1	9.5585	9.6397	9.6698	0.012
2	14.950	15.361	15.225	0.018
3	19.326	19.739	19.787	0.024
4	28.605	29.521	29.626	0.035
5	30.866	31.913	32.058	0.038

Table 4: Eigenvalue evaluation for L-shaped domain ($h = 1/32$)

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